A Developable Surface of Uniformly Negative Internal Angle Deficit

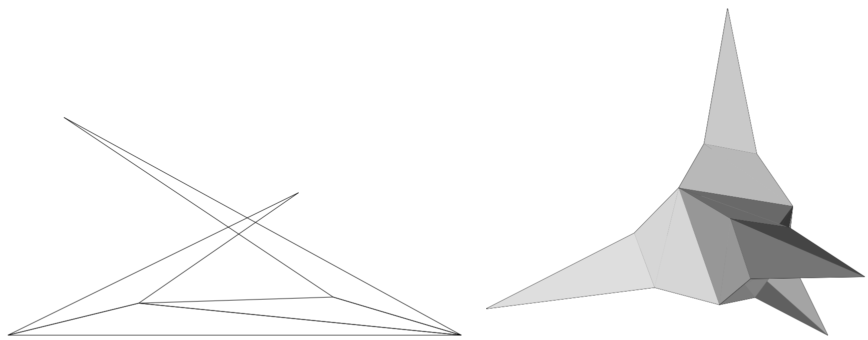
Phillips A Benton

Centre for Mathematical Sciences, Cambridge CB3 0WA, United Kingdom
A.Benton@damtp.cam.ac.uk

Abstract. The author addresses the Edge Unfolding Problem, the task of determining whether a given polyhedral mesh may be cut at its edges and developed into a planar polyhedral net. It is shown that simply-connected surfaces of negative interior curvature cannot be developed. An example is then given of a surface of negative interior curvature with two boundary loops, isomorphic to a cylinder, which is developable.

1 Introduction

The *Unfolding Problem* was formally posed as an open question by Shepherd in [S75], but unfoldings of polyhedral surfaces can be found as far back in published history as 1525 in the work of the German artist/mathematician Albrecht Dürer [S77] (pp. 316-347, 414-417). While a number of advances have refined the more general problem (Bern et al in [BDEK99], Benbernou et al in [BCO04], etc) the fundamental question of edge-unfolding remains unsolved: how to determine the set of edges of a polyhedral model to cut, allowing the model to unfold into a flattened, non-self-intersecting form.



(a) (b)

Fig. 1. (a) Namiki and Fukuda's overlapping tetrahedral unfolding; (b) an assembly of four instances of Bern et al's undevelopable Witch's Hat model

It is widely supposed that all convex polyhedra can be unfolded ([O98], p. 2). Small convex models have been identified which exhibit at least one illegal (self-intersecting) unfolding; Namiki and Fukuda [NF94] found the elegant example of the slim tetrahedron (Figure 1a.) No example has been found yet of a convex surface which admits no legal unfolding. In contrast, non-convex surfaces which cannot be unfolded are well-known, such as Bern et al's 'witch's hat' model [BDEK99] (Figure 1b.)

2 Definitions

A polyhedral surface is said to be *developable* if a subset of its edges can be found which may be cut such that the faces of the mesh remain connected by a net of edges about which the mesh may be unfolded, flattening to the plane without overlap.

An edge or series of edges broken in the course of developing a model is referred to as a *cut*.

Cuts join together to form the *cut graph*, a connected undirected graph of broken edges. It will be shown that there are restrictions on the number of loops in the cut graph, and on the number of loops which may be formed by taking the union of the cut graph with the graph of the boundary edges of the source mesh. A *branch* in the cut graph is any node of valency greater than two.

The dual of the cut graph is the *unfolding tree*, the undirected, connected, acyclic and planar graph of the connectivity of the net of faces which remains after all cuts have been made. The unfolding tree is a subset of the graph of connectivity of faces of the original source mesh.

The Gaussian curvature of the surface of any polyhedral mesh is zero everywhere except at the vertices, where it is infinite. A number of approaches have been devised for determining the discrete curvature of a surface at a vertex, and Meyer et al provide a very accessible description of the Gauss-Bonnet scheme in [MDSB03]. A simplified form of the Gauss-Bonnet scheme is the *angle deficit* method, well-described by Van Loon in [V94] (p.5). The angle deficit of a vertex V is 2π minus the sum of the corner angles of the polygonal facets meeting at V.

3 Cycles in the Cut Graph, Angle Deficit and Developability

By definition, the unfolding tree cannot possess cycles and must be connected. The cut graph may possess cycles; the circumstances in which this is permitted may be bounded explicitly:

Lemma 1 *The presence of a loop in the cut graph requires that there be at least one handle in the topology of the source mesh.*

Proof: A loop in the cut-graph of the surface forms a Jordan curve on the surface which separates any locally $\Re_2$ surface into exactly two discrete components[1]. If the source mesh is without handle (locally $\Re_2$) then the unfolding tree will be broken into two parts. Therefore either the surface must have a handle which connects the faces within the loop to the faces outside the loop; or, the cut-graph cannot loop.
□

Corollary 1 *The formation of a loop in taking the union of the cut graph with the set of all boundary edges of the source mesh requires that there be at least one handle in the topology of the source mesh.*

Proof: Boundary edges, like cut edges, have no dual in the unfolding tree. Loops in the (potentially disjoint) graph of boundary edges are permissible but, as in Lemma 1, new loops may not be introduced in taking the union of the cut graph and the boundary graph on a simply-connected surface. To do so would separate the surface into two or more disjoint parts as above.
□

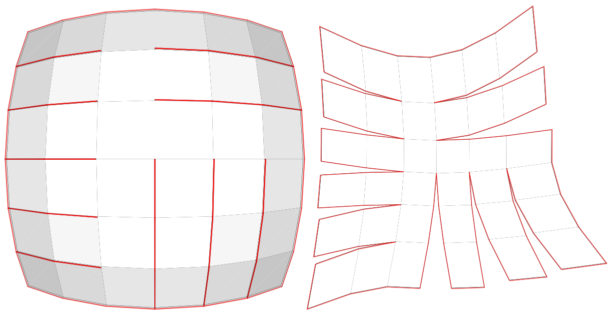


Fig. 2. Positive curvature: edge cuts terminate at inner vertices

There is a direct correspondence between the sign of the angle deficit at a vertex and the number of cuts it requires [P03]:

- At each spherical[2] vertex there must be at least one cut;
- At each hyperbolic[3] vertex there must be at least two cuts which take off one or more faces during the unfolding.

[1] From [G76], Appendix A.2, p.95. Griffiths does not prove this theorem directly, instead citing Newman's *Elements of the Topology of Plane Sets of Points*, Cambridge University Press, London (1954), p. 137.

[2] angle deficit greater than zero

[3] angle deficit less than zero

Vertices of zero angle deficit may be left completely untouched; they can flatten to the plane without cutting a single incident edge.

There can never be a leaf node (endpoint) of the cut graph on a vertex of negative angle deficit, because if at least two edges incident to a negative-curvature vertex must be cut then no such vertex can ever be a leaf node of the cut tree. If a branch of the cut-tree were to terminate at a negative-curvature vertex, only one incident edge would be broken.

This may be summarized as

Lemma 2 *No simply-connected surface of uniformly negative internal angle deficit is developable.*

> **Proof:** Suppose that there exists a simply-connected surface of uniformly negative internal angle deficit which is developable. To generate the unfolding, a cut graph must exist. This graph cannot loop, as the surface is simply connected; likewise, it may contain at most one vertex which lies on a boundary edge of the surface . If the graph does not loop then it must have at least two leaf nodes, where the edges of the cut graph originate and terminate, but all vertices have negative curvature. Therefore no such surface may exist.
> □

Corollary 2 *The number of branches in the cut-tree which develops a mesh of uniformly negative internal angle deficit cannot exceed $B - 2$, where B is the number of distinct boundary curves on the mesh.*

> **Proof:** The number of leaf nodes of a tree is two plus the sum of the valence minus one of each branch node in the tree. Thus each branch raises the required number of boundary curves by one. As above, there must be at most one leaf node per boundary curve and no leaf node may not fall on a boundary curve.
> □

Corollary 3 *The (topologically) simplest developable surface of uniformly negative internal angle deficit has at least two boundary curves.*

> **Proof:** It would be impossible for a graph to have -1 branch points.
> □

4 A Developable Surface of Uniformly Negative Internal Angle Deficit

A *coolinoid*[4] is the surface of rotation obtained by rotating a polynomial $f(t) = t^k + 1$ about the t axis. To obtain a discrete representation, the following parametric description is used:

```
  for (u:=0 to 1 step 1/du)
for (v:=-1 to 1 step 2/dv)
  x = cos(2πu) * (v^k + 1);
  y = v * h;
  z = sin(2πu) * (v^k + 1);
```

where $k \geq 1$ and $h > 0$. The model shown in Figure 3 was generated from ($h = 10.0$, $k = 2.2$, $du = 25$, $dv = 25$). Informally, h may be thought of as the 'stretch' of the model, k may be seen as the 'curviness' of the model and du and dv are the 'resolution' of the mesh.

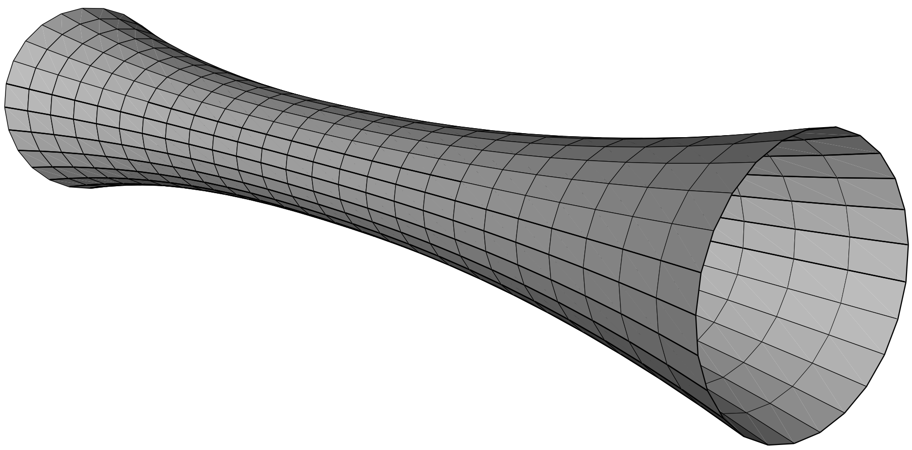


Fig. 3. Coolinoid

Many surfaces whose internal angle deficit is negative at every vertex are not developable, and several examples of such surfaces are shown in Figure 5. However, *there exist configurations of the coolinoid which are developable.* The model shown in Figure 3 unfolds into the spiraling net shown in Figure 4. The net has no self-intersection and is completely planar.

This demonstrates that

Theorem 1 *There exists a connected surface of uniformly negative internal angle deficit which is developable.*

[4] so named for its resemblance to the cooling tower of a power plant

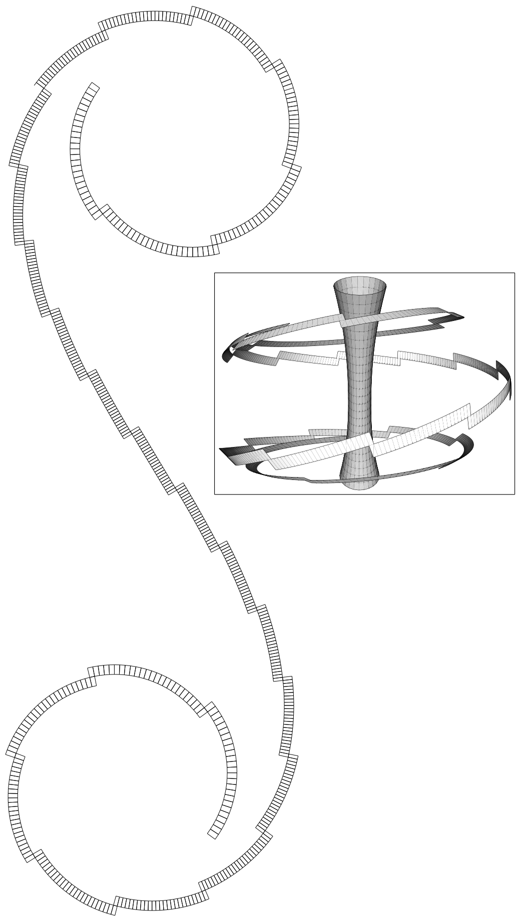


Fig. 4. Spiral unfolding of a coolinoid ($h = 10.0$, $k = 2.2$, $du = 25$, $dv = 25$.) Inset: The spiral unfolding in progress.

Proof: By example (Figure 4.)
Note: Numerical verification that the model shown in Figure 4 has no self-intersection may be obtained by verifying that Equations 1 and 2, below, have no common solution for $h = 10.0$, $k = 2.2$, $du = 25$, $dv = 25$.
□

5 A Black-Box Solution for Determining the Unfoldability of a Coolinoid

The observation that developability of the coolinoid varied as a function of h, k, du, dv spurred the development of a 'black-box' solution which could determine a

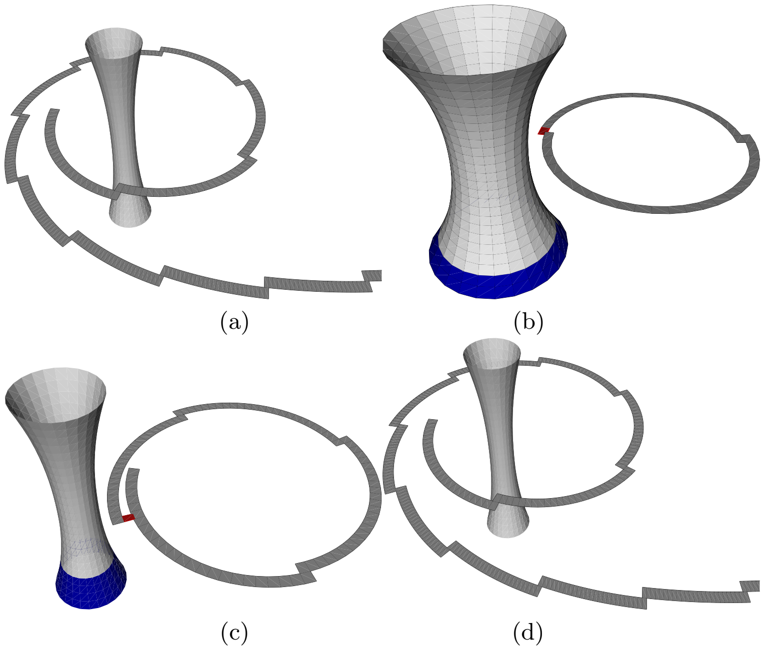


Fig. 5. Unfoldability is dependent on construction. Here k is fixed at 2.25, du and dv are fixed at 25, and $h = 1.2$, $h = 3.2$, $h = 6.2$ and $h = 9.2$. Only $h = 9.2$ is unfoldable.

priori whether a given coolinoid model could be unfolded. Note that for general non-convex polyhedra, there exists no known method faster than exhaustive sampling for determining unfoldability.

Corollary 2 allowed restriction of development to linear stripping heuristics. A series of heuristic schemes were evaluated and tested on a varied set of coolinoid models. The heuristic ultimately chosen was a 'spiral' unfolding, which unravels the model into a strip of consecutive faces ordered by adjacency in a counterclockwise traversal about the positive Y axis and then by ascending Y value. It was found that this scheme unfolded every model which could be unfolded by any other scheme plus a significant percentage more. No scheme was found which displayed a higher success rate.

Unwrapping a single row of faces from a coolinoid generates a curved arch of trapezoids which does not self-intersect; the flattened strip could never complete a full circle unless it already was one (which is the special case of the inner core of a torus, and undevelopable.) Each subsequent row, unrolled into a strip attached to the last face of the previous strip, will have less curve than the previous row until the unwrapping crosses the vertical midplane, after which the curved arch of each row will wrap symmetrically in the opposite direction. This creates a smooth spiral which unravels to a straight line and then curls back into another spiral.

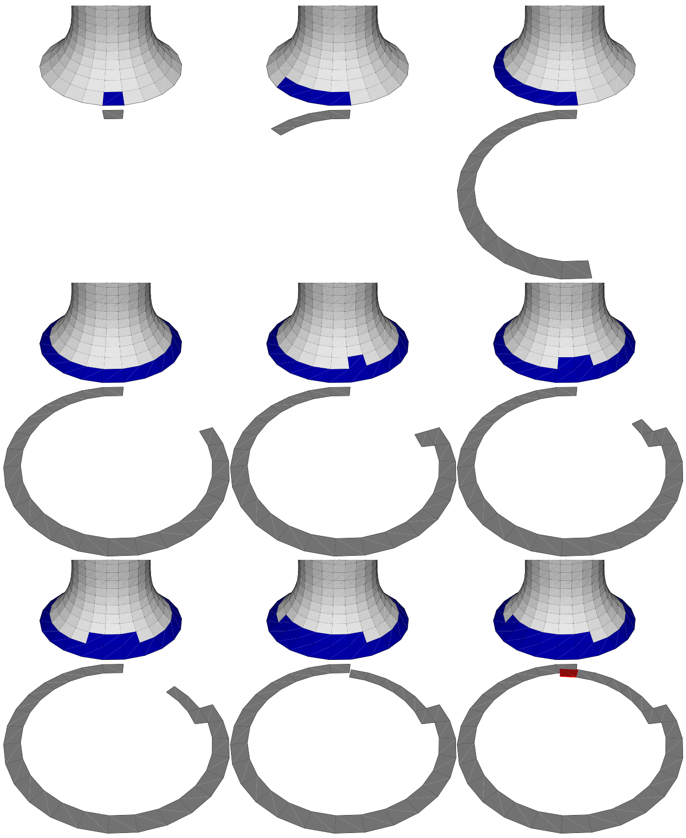


Fig. 6. Spiral unfolding with conflict in the second row

However, not all coolinoids can be spirally unfolded. The shared edge of the last face of a row i and the first face of row $i+1$ is the edge which lies *inside* the curl of the spiral; this moves each row inwards, towards the center of the spiral, by the height of the row. For many coolinoids this inwards step introduces a subsequent conflict between two rows, as shown in Figure 5a-c.

5.1 A Functional Expression of the Unfolding of a Coolinoid

The curve of the lower and upper borders of the spiral unfoldings of the faces of a coolinoid can be expressed as functions of a single linear parameter.

Taking h, k, du and dv as constants, define:

$$P(u,v) = \left[\left(cos\frac{2\pi u}{du} \right) \left(\left\| \frac{2v}{dv-1} \right\|^k + 1 \right), \frac{2vh}{dv-1}, \left(sin\frac{2\pi u}{du} \right) \left(\left\| \frac{2v}{dv-1} \right\|^k + 1 \right) \right]$$

$$Up(v) = P(0, v+1) - P(0, v)$$

$$Over(v) = P(1, v) - P(0, v)$$

$$\alpha(v) = cos^{-1}\left(\frac{Up(v)}{\|Up(v)\|} \cdot \frac{Over(v)}{\|Over(v)\|}\right)$$

$$\beta(v) = \pi - 2\alpha(v)$$

where

- $P(u, v)$ is the Coolinoid function. u, v range from 0 to $du - 1$ and $dv - 1$, respectively.
- $Up(v)$ is the step from the 'bottom' of row v to the 'top' of row v.
- $Over(v)$ is the step from one vertex at the bottom of row v to the next vertex in the row, ordering the vertices in a clockwise direction up the positive Y axis.
- $\alpha(v)$ is the acute inner angle of the trapezoidal faces of row v.
- $\beta(v)$ is the angle by which the unwrapping of row v will 'curl' in the plane at each face.

The following functions are then defined:

$$Turn(v) = \sum_{lvl=0}^{v} (du - 1) * \beta[lvl]$$

$$Hop(v, \theta) = \|Up(v)\| \, [cos(\theta), sin(\theta)]$$

$$Skip(v, \theta) = \|Over(v)\| \, [cos(\theta), sin(\theta)]$$

$$\begin{aligned} Jump(u, v) = {} & Hop(v - 1, (Turn(v - 1) + \alpha(v - 1) + \beta(v - 1))) + \\ & \sum_{i=1}^{u} Skip(v, Turn(v - 1) + i * \beta(v)) \end{aligned}$$

$$Outer(u, v) = Jump(u, v) + \sum_{lvl=0}^{v-1} Jump(du - 1, lvl)$$

$$Inner(u,v) = Outer(u,v) + Hop(v, Turn(v-1) + \alpha(v) + (u+1) * \beta(v)) + \lfloor u/(du-1) \rfloor * (Hop(v+1, Turn(v) + \alpha(v+1)) - Skip(v+1, Turn(v)))$$

such that

- $Turn(v)$ is the total curl introduced by the unwrapping of row v.
- $Hop(v, \theta)$ gives the vector $Up()$ rotated by θ.
- $Skip(v, \theta)$ gives the vector $Over()$ rotated by θ.
- $Jump(u, v)$ gives the offset of the unwrapping of the first u faces of row v.
- $Outer(u, v)$ give the position of the lower right corner of the unwrapping of the first $v-1$ rows and the first u faces of the v^{th} row.
- $Inner(u, v)$ gives the position of the upper right corner of the same faces as $Outer(u, v)$. The construction of $\lfloor \frac{u}{du-1} \rfloor$ is designed to shift the function back one face, allowing the first face of each row to share its lower edge with the last face of the row preceding.

The functions Outer() and Inner() are now defined over the range $\{u, v\}\ \varepsilon\ \{[0..du-1], [0..dv-1]\}$ with u and v both held integer. The unraveling unfolding evaluates these functions in a linear progression. Encapsulating this linear progression as

$$OuterRing(t) = Outer(\lfloor t \rfloor mod(du), \lfloor t/du \rfloor) \tag{1}$$

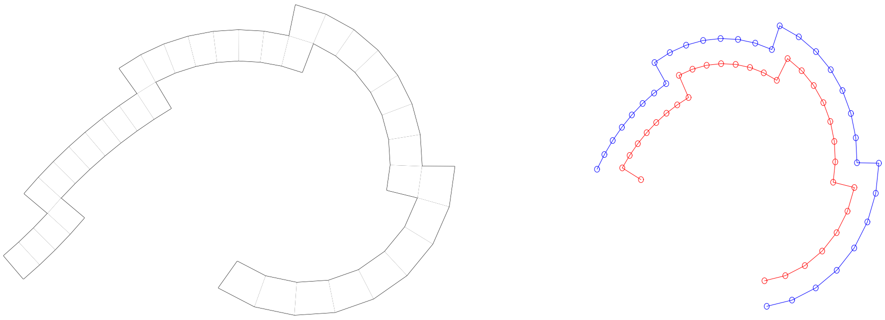
$$InnerRing(t) = Inner(\lfloor t \rfloor mod(du), \lfloor t/du \rfloor) \tag{2}$$

yields two univariate equations whose solution(s), if they exist, are the loci of intersection of the outer and inner border of the unfolding (Figure 7.) Testing for the unfoldability of a coolinoid is now reduced to solving for the intersection of these two equations.

6 Developability of the Coolinoid

As mesh resolution increases, the odds of the mesh being unfoldable decrease (Figure 8) echoing similar results gathered by J O'Rourke in Figure 2 of [O98]. The data shown in Figure 8 was gathered by taking the average developability over the range $\{h, k\}\varepsilon\{[0.5..14.75], [0..10]\}$ for each integer value of dim in the range [3..78], setting $du = dv = dim$.

Evaluating the developability of coolinoids continuously over the domain given above, an implicit surface is generated (Figure 9.) The surface shows extremely intriguing behavior. Predictably, it displays interleaved shelving effects in some areas, highlighting regions of $\{h, k, dim\}$ space where parity (odd/even) of the mesh dimension has an effect on the developability; this is most readily visible where $k < 1$. Such liminal regions are common in such surfaces. More



(a) (b)

Fig. 7. (a) An unfolding (b) The same unfolding evaluated in Mathematica(tm)

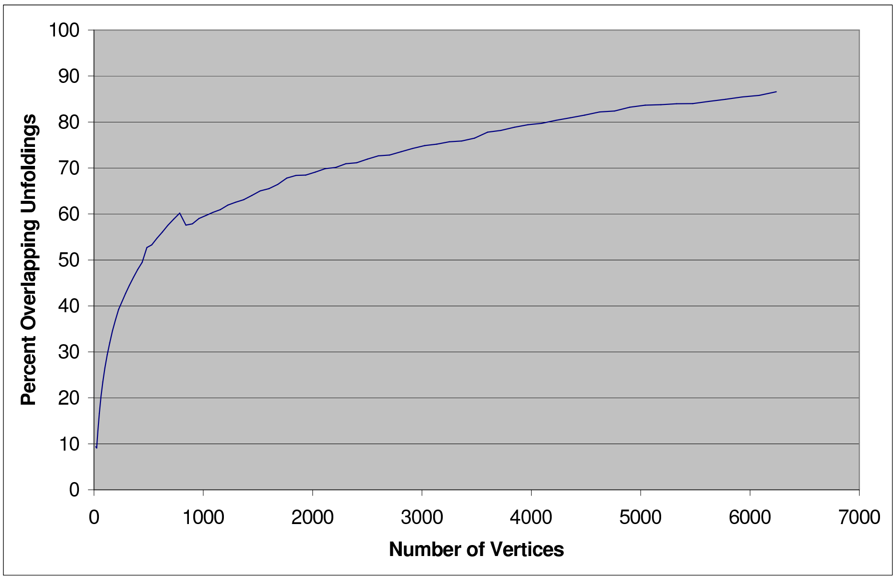


Fig. 8. Overlap vs Dimension

interestingly, an evolving wave pattern–hinting at fractal behavior–begins to appear in the isosurface as dimension increases. The wave hints at a much more complex isosurface than might be expected. Other ranges of symmetric and asymmetric behavior appear throughout the surface; further study is warranted.

In figure 9 the isosurface is shown looking from up the positive h axis. The positive k axis travels left-to-right and dim travels from the image's bottom to top. Note that the dim axis is integer, creating a voxel-like shelving effect (the 75 shelves in Figure 9) along the vertical axis.

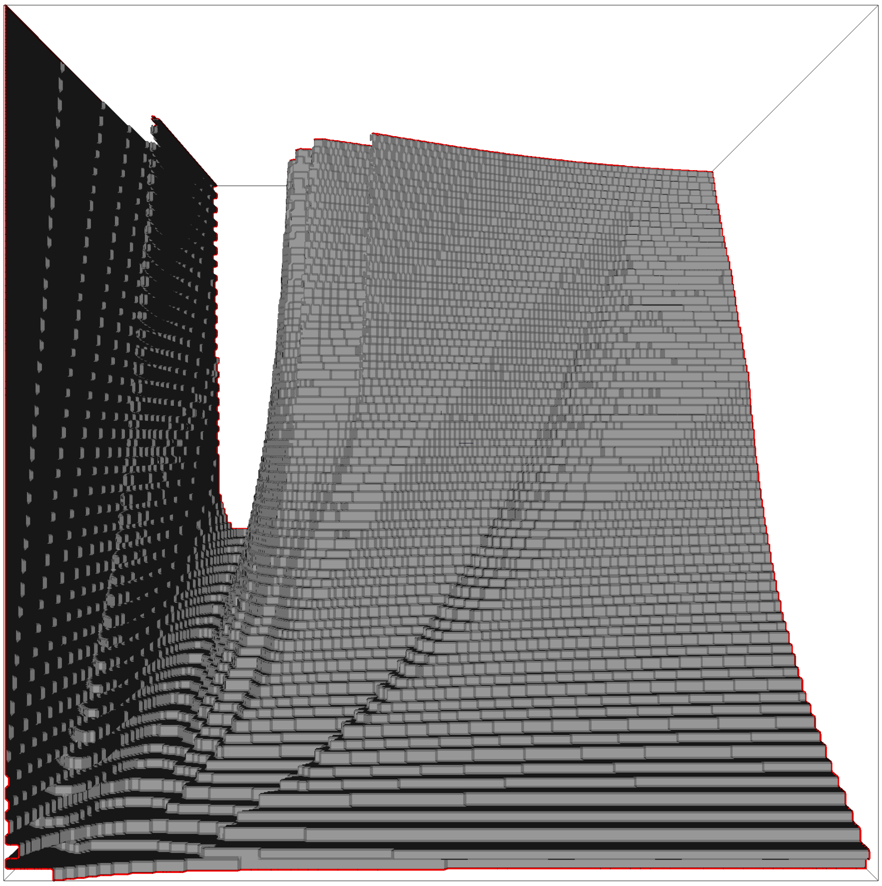


Fig. 9. Coolinoid unfoldability represented as an implicit surface in $\{h, k, dim\}$

Figure 10a shows the isosurface from the side, looking down the positive k axis towards the origin.

Figure 10b shows a detail of the lower h, k values. Note the interleaved parity-sensitive structure close to the origin in the $k \leq 1$ region, followed by a deep trough of undevelopability in the range $1 \leq k \leq 2$.

Figure 10c shows an overhead view, looking down on the model from the h axis. Note the wave pattern along the topmost border of the isosurface.

Figure 10d shows the model from above, looking directly down the positive dim axis. The wave in the isosurface is clearly visible.

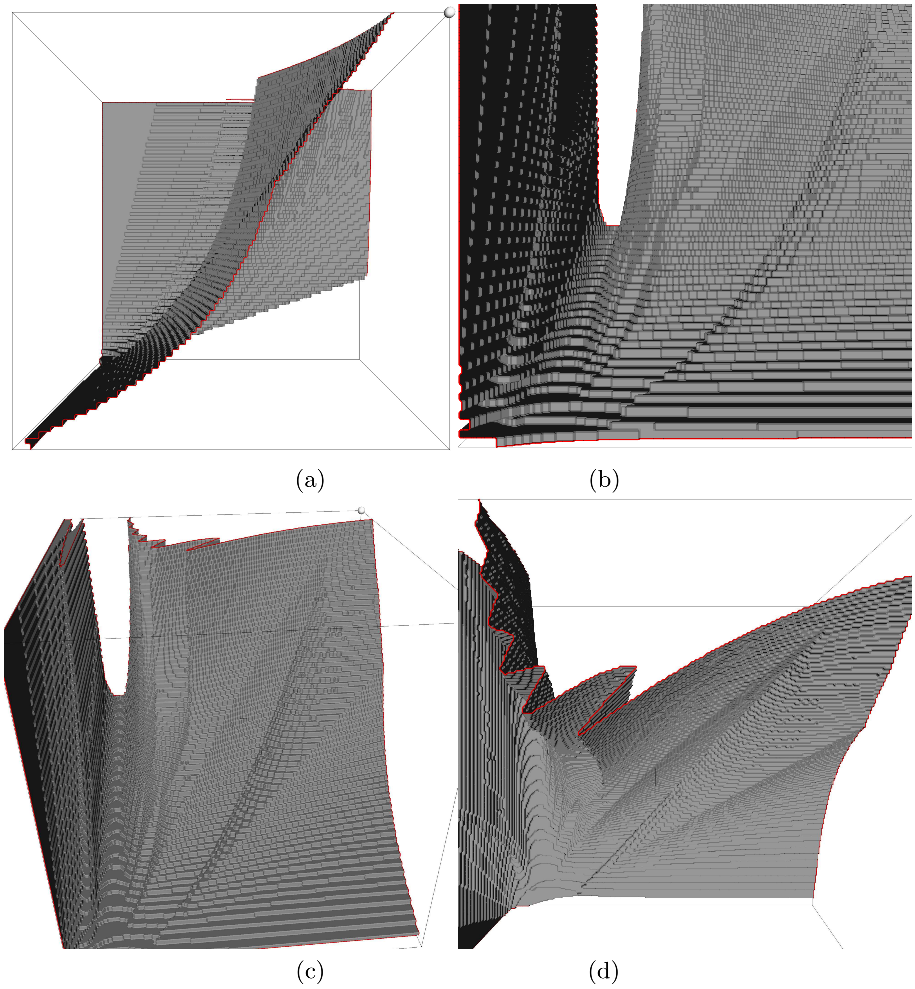


Fig. 10. Coolinoid unfoldability represented as an implicit surface in $\{h, k, dim\}$

7 Future Work

The coolinoid is very similar to functions such as $x^2 + y^2 - z^2 = 1$ and the *Catenoid*. Analysis comparable to that performed above would be quite instructive. The techniques demonstrated will apply to any surface which may be unfolded by the spiral unfolder.

A mathematically rigorous proof that the spiral unfolder was the optimal heuristic (ie., that there exists no coolinoid which is developable but which is not developed by the spiral unfolder) would be a significant advancement of the results presented.

The wave function which emerges in the upper ranges ($dim \rightarrow 50+$) of the coolinoid unfoldability isosurface displays fascinating fractal behavior which calls for ongoing investigation. J O'Rourke has suggested[5] that the wave is an artifact of the stepwise nature of the integer *dim* field in conjunction with h, a progression already becoming visible in Figure 5 a-d.

The interleaving effects for low-resolution models (Figure 10b, lower left) decay as *dim* rises. Does that decay flatten fully, or is it re-expressed at higher resolutions in the much subtler interleaved effects that appear at higher values of h?

In the isosurface shown, it was assumed that $du = dv$. It would be very interesting to decouple these two fields, plotting a four-dimensional isosurface, substituting one of the four axes for time and animating the evolution of the wave.

8 Conclusions

It has been shown that simply-connected surfaces of negative interior curvature cannot be unfolded. An example has been given of a developable surface of negative interior curvature with two boundary curves: the *coolinoid.* A black-box solution for determining the unfoldability of any given coolinoid has been found and further analysis of the developability of the coolinoid has yielded startlingly complex and intriguing results.

9 Acknowledgements

The author gratefully acknowledges the guidance and advice of Dr Malcolm Sabin, DAMTP, University of Cambridge.

References

[BCO04] N. Benbernou, P. Cahn, J. O'Rourke, Unfolding Smooth Prismatoids, Smith College Computer Science Technical Report 078, July 2004

[BDEK99] M. Bern, E. Demaine, D. Eppstein, E. Kuo, Ununfoldable Polyhedra, Proceedings of the 11th Canadian Conference on Computational Geometry (CCCG'99), Vancouver, British Columbia, Canada, August 15-18, 1999, pp 13–16

[G76] H. Griffiths, Surfaces, Cambridge University Press ISBN: 0 521 20696 0

[MDSB03] M. Meyer, M. Desbrun, P. Schroder, A. Barr, Discrete Differential Geometry Operators for Triangulated 2-Manifolds, VisMath '02 Proceedings

[NF94] M. Namiki, K. Fukuda, Unfolding 3-dimensional convex polytopes: A package for mathematica 1.2 or 2.0, *ftp://ftp.ifor.math.ethz.ch/pub/fukuda/mathematica/UnfoldPolytope.tar.Z*

[O98] J. O'Rourke, Folding and unfolding in computational geometry, Lecture Notes Comput. Sci.. Vol. 1763, Springer-Verlag, Berlin, 2000, pp 258–266

[5] Personal communications, February 2007

[P03] K. Polthier, Imaging maths - Unfolding polyhedra, *http://plus.maths.org/issue27/features/mathart/index.html*

[S75] G. Shephard, Convex polytopes with convex nets, Math. Proc. Camb. Phil. Soc, 1975

[S77] W. Strauss, The Painter's Manual by Albrecht Durer (1525), Abaris Books (June, 1977) ISBN: 0913870528

[V94] B. Van Loon, *Geodesic Domes*, Tarquin Press (Feb 1994) ISBN: 0906212928